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Lie algebra invariant adjoint fermion systems with $N = 2$ supersymmetry

A J Macfarlane

Centre for Mathematical Sciences, DAMTP, Wilberforce Road, Cambridge CB3 0WA, UK

E-mail: a.j.macfarlane@damtp.cam.ac.uk

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Abstract

This paper describes supersymmetric quantum mechanical models $\mathcal{S}(D)$ of D fermions invariant under the adjoint action of a compact simple Lie algebra $\mathfrak{g}$ of dimension D. It determines the spectrum, ground state properties and the full Fock space $\mathcal{F}$ structure of such interacting models. It also discusses hidden supersymmetries, partner theories and spectrum generating algebras for the models $\mathcal{S}(D)$.

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1. Introduction

Supersymmetry in general and supersymmetric quantum mechanics (SQM) in particular have been matters of enduring interest, in which however the search for new insights may still usefully be pursued. An aspect of SQM that has featured in our previous work [1] and is central to the present paper is the study of purely fermionic models in SQM. We would like to think of this as of interest in its own right, and as a means of providing tools of use in the construction of the fermionic sectors of more general models in SQM. The study of purely fermionic models does indeed show up features of intrinsic interest, only some of which are shared with more general models. One shared feature concerns the existence of hidden supersymmetries in SQM, and the emergence of partner theories for the ones originally formulated. Early work in this general area includes [2–4] where particle motion in background fields such as those provided by notable solutions of Einstein's equation is studied. A more recent paper [5] reviews the appearance of hidden supersymmetries in various models in SQM.

Suppose, in a given theory, there is found, for example from a superspace formulation, what we will call a natural (or built in) supersymmetry with supercharge Q such that

$$Q^2 = 0 \qquad Q^{\dagger 2} = 0 \qquad \{Q, Q^\dagger\} = 2H_q \tag{1}$$

where H_q is the Hamiltonian. Then in some cases we may be able to infer the existence of a hidden supersymmetry, one not explicitly built into our theory, with a supercharge Q'. If this supercharge obeys

$$\{Q, Q'\} = 0 \qquad \{Q, Q'^{\dagger}\} = 0 \tag{2}$$

its existence is compatible with the natural supersymmetry, and we also have

$$Q'^2 = 0 \qquad Q'^{\dagger 2} = 0 \qquad \{Q', Q'^{\dagger}\} = 2K. \tag{3}$$

Here K is some Hermitian operator for which we usually have an obvious interpretation. If $K = H_q$ then we have simply promoted the supersymmetry of the original formulation of the theory from $N = 1$, say, to $N = 2$ supersymmetry, and should perhaps take account of this by trying to generalize the superfield approach followed initially. But other obviously interesting possibilities are well known; see, e.g., [6] for the case of motion of a particle of spin one-half in the field of a Dirac monopole, or [7] which involves the background field of a Wu–Yang monopole. Possibilities of the types so far mentioned are illustrated also in suitable variants of the purely fermionic models of [1]. Again see also [5]. When $K \neq H_q$, we may reformulate our original theory so as to define a partner theory: the latter has K as its Hamiltonian and Q' as the generator of its primary supersymmetry, with Q now seen merely in the secondary role of generator of a hidden or additional supersymmetry. In the Dirac monopole case K is to within a constant $\mathbf{J}^2$, where $\mathbf{J}$ is the total angular momentum operator of the models, and the partner theory describes motion confined to the sphere S^2 in the presence of the Dirac monopole field.

A class of purely fermionic models of SQM that is of good potential for our studies involves a set of D fermions which transform according to the adjoint representation *ad* of a compact simple Lie algebra $\mathfrak{g}$, $\dim \mathfrak{g} = D$. Such models are of value because they can be fully analysed and will be seen to exhibit some remarkable features. Moreover, much of the information gathered is of potential use in more general $\mathfrak{g}$-invariant models, whose fermionic sector involves adjoint fermions, e.g., the fermionic Fock space structure discussed below.

In this paper, we study $N = 2$ supersymmetric models $\mathcal{S}(D)$ of systems of D fermions which transform under the compact simple Lie algebra $\mathfrak{g}$ according to its adjoint representation. Models of the type $\mathcal{S}(D)$ have been treated previously [5], but the analysis is carried further here. The models $\mathcal{S}(D)$ possess a hidden supersymmetry generator Q', which is compatibly related to that of the built- in supersymmetry as in (2). But at this point a remarkable feature emerges. The operator K of (3) is related to H_q by

$$K = \tfrac{1}{3}D - H_q = \mathcal{C}^{(2)} \tag{4}$$

where $\mathcal{C}^{(2)}$ denotes the quadratic Casimir operator of $\mathfrak{g}$. Such a feature has not been found in any previously analysed model. It has the curious effect that the spectrum of K is inverted relative to that of H_q. However, the spectrum of energies of each of H_q and K is confined to the interval $0 \leqslant E \leqslant \frac{1}{3}D$ of the real line. Further, the partner theory $\mathcal{P}(D)$ of $\mathcal{S}(D)$ has as its two *ad*-invariant ground states of energy zero the fermionic Fock vacuum, and the completely filled state. On the other hand the ground states of $\mathcal{S}(D)$ itself, i.e. of H_q, which are the highest E states of $\mathcal{P}(D)$, have a different but notable description, one that is valid for all $\mathfrak{g}$. For any $\mathfrak{g}$, the ground states in question belong to the irrep $\Lambda = (2, 2, \ldots, 2)$ of $\mathfrak{g}$, where we use standard highest weight notation, for which $\mathcal{C}^{(2)}$ has the eigenvalue

$$c_2(2, 2, \ldots, 2) = \tfrac{1}{3}D \tag{5}$$

so that H_q has zero eigenvalue. Thus Λ provides the ground states of H_q, occurring 2^l times in all, so that the total ground state degeneracy is

$$2^l \dim(2, 2, \ldots, 2) = 2^l\, 3^{(D-l)/2} \tag{6}$$

where $l = \text{rank}\,\mathfrak{g}$. Some detailed work in the representation theory of Lie algebras, outlined in section 7, is needed to establish the general results of this paragraph.

One might at this point very well ask if we should place the partner theory $\mathcal{P}(D)$ in the primary role, viewing H_q and Q as of secondary importance. One is motivated to take such a view by the fact that the uninverted spectrum of K is physically more sensible than that of H_q. However, one can insist that $\mathcal{S}(D)$ is more fundamental because we have a natural route towards its formulation via superfield methods, whereas no such approach to the partner theory is known to us at present.

Our study of the models $\mathcal{S}(D)$ generates much other information of significance. We are able to find the complete spectrum of the $\mathcal{S}(D)$, and hence of the partner theory, and a complete description of all the corresponding eigenstates of H_q and of K. In other words we can obtain a complete description of the Fock space $\mathcal{F}(D)$ of our models. Since the $D = \dim\mathfrak{g}$ fermions of $\mathcal{S}(D)$ transform according to the adjoint representation ad of $\mathfrak{g}$, this means that, to identify the states of each of the n-fermion subspaces $\mathcal{F}_n$ of $\mathcal{F}(D)$, we must decompose into irreducible representations of $\mathfrak{g}$ the totally antisymmetrized n-fold direct product $ad^{\wedge n}$ for each $n \in \{1, 2, \ldots, D = \dim\mathfrak{g}\}$. This can be done easily from first principles for small $\mathfrak{g}$, and in general by computer program. We present results below in the form of tables for $\mathfrak{g} = a_1, a_2, b_2, g_2, a_3$. The results for a_1 are elementary; those for a_2 were found previously by tensorial means [8]; the other cases were treated first by group theoretic methods, and confirmed by a computer program, which also provided the dimensions and Casimir eigenvalues of each representation of each $\mathfrak{g}$ needed. The data presented should be of use beyond the context of this paper. The cases b_3, c_3, a_4 have also been fully analysed, but the data have not been included because of the size of the tables that would be needed.

The material of the paper is organized as follows. Section 2 reviews the formulation of the models $\mathcal{S}(D)$, their hidden supersymmetries and partner theories $\mathcal{P}(D)$. Section 3 presents and comments on data for $\mathfrak{g}$ of rank $l \leqslant 2$. This includes the listing for all relevant n of the irrep content of each n-fermion subspace of the Fock space of each model. Section 4 explains the results that apply to the ground states of models $\mathcal{S}(D)$ and $\mathcal{P}(D)$ for any $\mathfrak{g}$; however the crucial proofs are deferred to section 7. Section 5 considers additional features that may emerge for models with $\mathfrak{g}$ of rank higher than 2, when higher order ad-invariant fermionic operators enter the picture, illustrating briefly with reference to a_3. Section 6 identifies the spectrum generating algebra for $\mathcal{S}(D)$ invariant under any compact simple $\mathfrak{g}$.

Section 7 discusses the decomposition into irreps of $ad^{\wedge n}$ of $\mathfrak{g}$, providing, as briefly as we found possible, background essential for understanding the general results of section 4 for ground states.

1.1. Notation and convention

Let $\mathfrak{g}$ be a compact simple Lie algebra with generators X_k such that

$$[X_i, X_j] = \mathrm{i}c_{ijk}X_k. \tag{7}$$

Its adjoint representation ad is defined by $X_k \mapsto ad_k$

$$(ad_i)_{jk} = -\mathrm{i}c_{ijk} \tag{8}$$

and our normalizations are fixed by requiring that the Cartan–Killing form of $\mathfrak{g}$ satisfies

$$\mathrm{Tr}(ad_j ad_k) = c_{pqj}c_{pqk} = \delta_{jk}. \tag{9}$$

It follows that the quadratic Casimir operator of $\mathfrak{g}$

$$C^{(2)} = X_k X_k \tag{10}$$

has, for each $\mathfrak{g}$, the eigenvalue

$$c^{(2)}(ad) = 1. \tag{11}$$

We use the informal abbreviation irrep for irreducible representation.

2. The *D*-fermion system

Following [5] we may employ a method based on chiral fermionic $N = 2$ superfields [1] to reach a Lagrangian

$$L = \tfrac{1}{2}\mathrm{i}(\mu_i^*\dot{\mu}_i + \mu_i\dot{\mu}_i^*) - 2J_iJ_i^* \tag{12}$$

where $J_i = -\frac{1}{2}\mathrm{i}C_{ijk}\mu_j\mu_k$, involving the structure constants of a Lie algebra of dimension $\dim\mathfrak{g} = D$. The Lagrangian L is manifestly $\mathfrak{g}$-invariant. The canonical Dirac bracket relations are

$$\{\mu_i, \mu_j^*\}_D = -\mathrm{i}\delta_{ij} \qquad \{\mu_i, \mu_j\}_D = 0 \qquad \{\mu_i^*, \mu_j^*\}_D = 0 \tag{13}$$

and the classical Hamiltonian is

$$H_{cl} = 2J_iJ_i^*. \tag{14}$$

The $N = 2$ supersymmetry of the theory follows from the fact that the supercharges

$$Q = \tfrac{1}{3}\mathrm{i}C_{ijk}\mu_i\mu_j\mu_k \qquad Q^* = \tfrac{1}{3}\mathrm{i}C_{ijk}\mu_i^*\mu_j^*\mu_k^* \tag{15}$$

generate canonically the supersymmetry transformations that leave the action of the theory invariant, and obey

$$\{Q, Q^*\}_D = 2H_{cl} \qquad \{Q, H_{cl}\}_D = 0. \tag{16}$$

In the passage to the quantum theory, we write $\mu_i = c_i$ and $\mu^\dagger = \pi_i$ to denote the fermion creation and annihilation operators, and impose the anticommutation relations

$$\{c_i, \pi_j\} = \delta_{ij}. \tag{17}$$

Also the quantum Hamiltonian is given by

$$H_q = \tfrac{1}{2}\{Q, Q^\dagger\} = \left\{J_i, J_i^\dagger\right\} - \tfrac{1}{6}D. \tag{18}$$

We expect H_q to be closely related to the quadratic Casimir operator of $\mathfrak{g}$ (10), and, by direct calculation using the Jacobi identity and (9), we find

$$2J_iJ_i^\dagger = N - X_iX_i \qquad 2J_i^\dagger J_i = D - N - X_iX_i \tag{19}$$

where X_i has the representation

$$X_i = -\mathrm{i}c_{ijk}c_j\pi_k \qquad \text{and} \qquad N = c_i\pi_i \tag{20}$$

is the fermion number operator. Hence

$$\left[J_i, J_i^\dagger\right] = \tfrac{1}{2}D - N \tag{21}$$

and

$$H_q = \tfrac{1}{3}D - X_iX_i. \tag{22}$$

Despite its appearance H_q is a positive definite operator. This follows (18). Further, the analysis of section 4 shows, by exhibiting them, that eigenstates of H_q of zero energy exist for each simple compact $\mathfrak{g}$.

Table 1. $\mathcal{F}$ for a_1.

States	n	Irreps	Energy E
$\|0\rangle$	0	1	1
$c_i\|0\rangle$	1	3	0
$\epsilon_{ijk}c_jc_k\|0\rangle$	2	3	0
$Q\|0\rangle$	3	1	1

2.1. Hidden supersymmetries

It is natural in the context to consider the $\mathfrak{g}$-invariant fermionic operators

$$Q' = q_{21} = \tfrac{1}{2}\mathrm{i}C_{ijk}c_jc_j\pi_k \qquad Q'^{\dagger} = q_{12} = \tfrac{1}{2}\mathrm{i}C_{ijk}c_i\pi_j\pi_k. \tag{23}$$

These operators each anticommute with each of $Q = q_{30}$ and $Q^{\dagger} = q_{03}$, and obey

$$\{q_{21}, q_{12}\} = \tfrac{1}{2}X_iX_i \qquad q_{21}^2 = 0 \qquad q_{12}^2 = 0. \tag{24}$$

It follows that q_{21} and q_{12} commute with X_iX_i and with H_q, so that they generate a hidden supersymmetry of the original theory.

We may regard

$$K = H_p = X_iX_i \tag{25}$$

as itself defining the Hamiltonian of a theory in which Q' generates the natural supersymmetry, and Q a hidden one. This theory is the partner theory to the original one. We do not have a Lagrangian formulation for it but it will be seen, from the results of section 4, to have the nice features that its ground states are the Fock vacuum and the completely filled state, and that its highest energy eigenvalue is $D/3$.

3. Data for $a_1 = su(2)$, $b_2 = so(5)$, g_2 and $a_3 = su(4)$

3.1. $a_1 = su(2)$

Here $D = 3$, and the Fock space $\mathcal{F}$ of the corresponding model contains 2^3 basis states, shown in table 1.

The triplets are the ground states, of lower energy than the $su(2)$-invariant states. Also energies are independent of the fermion number of a state, depending only on the eigenvalue of $\mathcal{C}^{(2)}$. Similar features are present in models for other simple compact $\mathfrak{g}$.

The a_1 model is very simple but it nevertheless reflects the consequences of supersymmetry that are more richly illustrated in models based on larger $\mathfrak{g}$. The triplet states form a doublet from the standpoint of supersymmetry

$$Q'c_i|0\rangle = \epsilon_{ijk}c_jc_k|0\rangle. \tag{26}$$

Since Q' is $su(2)$-invariant, this accounts for the fact that the two triplets have the same energy despite having different fermion numbers. Likewise the two singlets form a supersymmetry doublet, this time with the aid of Q, cf (15), as the $n = 3$ entry in table 1 itself indicates.

3.2. $a_2 = su(3)$

Here $D = 8$ and the construction of the basis states of the Fock space of the a_2 model was carried out by tensorial means in [8].

Table 2. Data for a_2.

$d(\lambda, \mu)$	(λ, μ)	$c_2(\lambda, \mu)$	E_q
1	(0, 0)	0	$\frac{8}{3}$
8	(1, 1)	1	$\frac{5}{3}$
20	$(3, 0) \oplus (0, 3)$	2	$\frac{2}{3}$
27	(2, 2)	$\frac{8}{3}$	0
64	(3, 3)	5	
70	$(4, 1) \oplus (1, 4)$	4	

Table 3. $\mathcal{F}$ for a_2.

n	dim $ad^{\wedge n}$	1	8	20	27
0(8)	1	1			
1(7)	8		1		
2(6)	28		1	1	
3(5)	56	1	1	1	1
4	70		2		2

Table 2 gives some data for a_2, for use here and in section 8. Table 3 gives a listing of the irreps of a_2 involved in the Fock space of the a_2 model $\mathcal{S}(8)$. The energies E_q of the states of $\mathcal{S}(8)$ occurring in the Fock space are given in table 2. Here E_q is the eigenvalue for (λ, μ) of the Hamiltonian H_q of (18):

$$E_q = \tfrac{1}{3}D - c_2(\lambda, \mu). \tag{27}$$

The nature of the zero energy ground states conforms to the general arguments of section 4. The explicit construction of all the states, the analogue of the information explicitly present in table 1, is to be found in [8], but we do not need it here. We note the reference to the irrep 20, which is an irrep only over the real number field. Over the complex numbers $20 \equiv 10 \oplus \overline{10}$.

To describe the supersymmetry structure, we employ the somewhat loose notation $|R, n\rangle$ for some linear combination of the states of the irrep R of a_2 of fermion number n.

For the case $R = 8$, we have

$$Q'|8, 1\rangle = |8, 2\rangle \qquad Q'|8, 4\rangle = |8, 5\rangle \tag{28}$$

and

$$Q|8, 1\rangle = |8, 4\rangle \qquad Q|8, 2\rangle = |8, 5\rangle \tag{29}$$

as well as

$$QQ'|8, 1\rangle = |8, 5\rangle. \tag{30}$$

Thus we say there is a supersymmetry quadruplet of $R = 8$ states with $n = 1, 2, 4, 5$, and in addition another quadruplet in an evident sense Hodge-dual to it, with $n = 3, 4, 6, 7$. This involves states $|8, 4'\rangle$ orthogonal to the states $|8, 4\rangle$ of (28), (29). Similarly there is a self-Hodge-dual quadruplet of $R = 20$ states, and doublets with $R = 1$ and $R = 27$, together with the duals of each.

The tensorial methods of [8] allow completely explicit versions of the statements of the last paragraph to be made. For example, the states $|8, 1\rangle$, $|8, 2\rangle$ correspond to the adjoint

Table 4. Data for b_2.

$d(\lambda,\mu)$	(λ,μ)	$c_2(\lambda,\mu)$	E_q	$d(\lambda,\mu)$	(λ,μ)	$c_2(\lambda,\mu)$	E_q
1	(0, 0)	0	$\frac{10}{3}$	35′	(0, 4)	$\frac{8}{3}$	$\frac{2}{3}$
5	(1, 0)	$\frac{2}{3}$	$\frac{8}{3}$	81	(2, 2)	$\frac{10}{3}$	0
10	(0, 2)	1	$\frac{7}{3}$				
14	(2, 0)	$\frac{5}{3}$	$\frac{5}{3}$	105	(1, 4)	4	
30	(3, 0)	3	$\frac{1}{3}$	84	(0, 6)	5	
35	(1, 2)	2	$\frac{4}{3}$	154	(3, 2)	5	

Table 5. $\mathcal{F}$ for b_2.

n	dim $ad^{\wedge n}$	1	5	10	14	30	35	35′	81
0(10)	1	1							
1(9)	10			1					
2(8)	45			1			1		
3(7)	120	1	1		1	1	1	1	
4(6)	210		1	1	1	1	1	1	1
5	252			2			2		2

vectors $c_i, Q'c_i$. Using standard $su(3)$ notation, $Q'c_i \propto d_i, d_i = f_{ijk}c_jc_k$. The vectors $e_i = d_{ijk}c_jd_k$, $Q'e_i$ correspond to $|8,3\rangle$, $|8,4\rangle$, while the action of Q on all these produces the remaining states, with, in particular, $|8,4\rangle'$ corresponding to Qc_i, which is linearly independent of $|8,4\rangle$. We note that $f_{ijk}c_jd_k$ is absent because the Jacobi identity implies that it vanishes.

We might just mention that one can pass from $|1,0\rangle$ to $|1,5\rangle$ using an $su(3)$-invariant supercharge fifth order in fermionic variables, but we will leave systematic study of such issues to our work on the a_3 model.

3.3. $b_2 = so(5)$

The tensorial method of attacking the structure of the Fock space $\mathcal{F}$ of the b_2 model is of marginal viability and then definitely unattractive, so we turn to alternative approaches. Any such approach depends on the fact that the subspace $\mathcal{F}^n$ of $\mathcal{F}$ with fermion number $N = n, n \in \{0, 1, \dots, D = \dim \mathfrak{g}\}$, carries the representation

$$ad^{\wedge n} = \underbrace{ad \wedge ad \wedge \cdots \wedge ad}_{n \text{ factors}} \tag{31}$$

of $\mathfrak{g}$. An elementary method of finding the decomposition of $ad^{\wedge n}$ into irreps for all n for sufficiently small $\mathfrak{g}$ is outlined in section 7. When this approach was applied to a_2, it confirmed previous results [8].

We present next some data about the irreps of b_2, in table 4, followed by table 5, which exhibits the irrep content of each $\mathcal{F}^n$.

We note that only tensorial irreps of b_2 arise in the decompositions of $ad^{\wedge n}$ for b_2. These all have even μ, whereas spinorial irreps have odd μ-values.

Table 5 accounts for all the $2^{10} = 1024$ states of $\mathcal{F}$. The supersymmetry multiplet structure—doublets and quadruplets—reflects the same properties as were seen for a_2: one $R = 1$ doublet, one quadruplet for each of $R = 10$ and $R = 35$, together with Hodge duals of each of these; one self-dual quadruplet for each of $R = 5, 14, 30, 35$, and finally one doublet

Table 6. Data for g_2.

$d(\lambda, \mu)$	(λ, μ)	$c_2(\lambda, \mu)$	E_q	$d(\lambda, \mu)$	(λ, μ)	$c_2(\lambda, \mu)$	E_q
1	(0, 0)	0	$\frac{14}{3}$	273	(3, 0)	$\frac{9}{2}$	$\frac{1}{6}$
7	(0, 1)	$\frac{1}{2}$	$\frac{25}{6}$	286	(2, 1)	$\frac{7}{2}$	$\frac{7}{6}$
14	(1, 0)	1	$\frac{11}{3}$	378	(0, 5)	$\frac{25}{6}$	$\frac{1}{2}$
27	(0, 2)	$\frac{7}{6}$	$\frac{7}{2}$	448	(1, 3)	$\frac{15}{4}$	$\frac{11}{12}$
64	(1, 1)	$\frac{7}{4}$	$\frac{35}{12}$	729	(2, 2)	$\frac{14}{3}$	0
77	(2, 0)	$\frac{5}{2}$	$\frac{13}{6}$				
77′	(0, 3)	2	$\frac{8}{3}$	924	(1, 4)	5	
182	(0, 4)	3	$\frac{5}{3}$	1547	(2, 3)	6	
189	(1, 2)	$\frac{8}{3}$	2				

Table 7. $\mathcal{F}$ for g_2.

n	dim $ad^{\wedge n}$	1	7	14	27	64	77	77′	182	189	273	286	378	448	729
0(14)	1	1													
1(13)	14			1											
2(12)	91			1				1							
3(11)	364	1			1		1	1	1						
4(10)	1001			1	1	1	1		1	1				1	
5(9)	2002		1	1		1		2		2	1	1	1	1	
6(7)	3003		1		1	1	1	2	1	2	1	1	1	1	1
8	3432				2	2	2		2	2				2	2

for $R = 81$ plus its dual. The ground states of the system are again provided by the highest irreps, $R = (2, 2) = 81$ with $c_2(R) = \frac{10}{3}$ so that (22) gives $E = 0$.

3.4. g_2

Data for g_2 are in table 6, and the Fock structure can be read off table 7. The supersymmetry multiplet structure is qualitatively as for the smaller rank two Lie algebras, only more complicated. For example, for $R = 77' = (0, 3)$, we find three quadruplets, one self-Hodge-dual, the other pair dual to each other. Also for $R = 189 = (1, 2)$, one gets a quadruplet and a doublet, plus the duals.

Again the irrep $R = (2, 2) = 729$ provides the zero energy ground states of the system, since $c_2(729) = \frac{14}{3}$ and $D = \dim g_2 = 14$.

4. The ground states for any simple compact $\mathfrak{g}$

The method outlined in the appendix for the decomposition of $ad^{\wedge n}$ tells that at the stage at which $n = \frac{1}{2}(D-l)$, the highest weight is the sum of all the positive roots, $2\delta = (2, 2, \ldots, 2)$. So an irrep with this highest weight certainly occurs in the decomposition of $ad^{\wedge n}$, and hence in the Fock space of the models $\mathcal{S}(D)$. In fact no irrep of higher highest weight occurs, and, for $\mathfrak{g}$ of rank l, the irrep of highest weight 2δ occurs 2^l times in the Fock space of $\mathcal{S}(D)$. Proof of this is indicated in section 7. It can be seen that the data in the tables agree with the claim.

We now use well-known formulae [9] for the dimension $\dim(\Lambda)$, and the eigenvalue $c_2(\Lambda)$ of the quadratic Casimir operator of an irrep of $\mathfrak{g}$ of highest weight Λ, namely

$$\dim(\Lambda) = \Pi_{\text{pos.roots}} \left(1 + \frac{\Lambda.\alpha}{\delta.\alpha}\right) \tag{32}$$

and

$$c_2(\Lambda) = (\Lambda, \Lambda + 2\delta). \tag{33}$$

For our irrep of highest weight 2δ, these formulae give

$$\dim(2\delta) = \dim(2, 2, \ldots, 2) = 3^{(D-l)/2} \tag{34}$$

and

$$c_2(2\delta) = c_2(2, 2, \ldots, 2) = 8(\delta, \delta) = \tfrac{1}{3} c_2(ad) D = \tfrac{1}{3} D. \tag{35}$$

In this last equation we have used the strange formula [10], p 119, to evaluate (δ, δ), and also the result $c_2(ad) = 1$ of our normalization convention (9).

It now follows from (22) for H_q and (35) that the states of the irreps $2\delta = (2, 2, \ldots, 2)$ provide zero energy ground states of $\mathcal{S}(D)$ for each simple compact Lie algebra $\mathfrak{g}$. Further the *ad*-invariant Fock vacuum and the completely filled Fock space state, also *ad*-invariant, are the states of highest energy $D/3$ of H_q. They are, however, the ground states of the partner theories $\mathcal{P}(D)$.

We have already commented (a) that the spectrum of the $\mathcal{P}(D)$ is inverted with respect to that of the original model $\mathcal{S}(D)$ and thus of more physically natural appearance, and (b) on the point of view of $\mathcal{P}(D)$ *vis-á-vis* $\mathcal{S}(D)$ that this affords.

5. The rank three Lie algebra $a_3 = su(4)$

5.1. Irrep and Fock space data

The following tables give information about irreps of a_3, and about the Fock space $\mathcal{F} = \sum_{n=0}^{15} \mathcal{F}_n$. Only irreps (λ, μ, ν) for which $\lambda + \nu$ is even occur in the analysis of $\mathcal{S}(D)$ for a_3. In other words only the irreps of even quadrality of a_3 [9] occur; these are the tensorial irreps of d_3, which is isomorphic to a_3. Also (λ, μ, ν) is self-conjugate if and only if $\lambda = \nu$, and table 8 lists only one member of any conjugate pair of representations. Also all the irreps of a_3 irreducible over the reals are either self-conjugate or else are the direct sums of conjugate pairs. In table 9, for the sake of brevity, we have written

$$70 = 35 \oplus \overline{35} \qquad 90 = 45 \oplus \overline{45} \qquad 512 = 256 \oplus \overline{256} \qquad 560 = 280 \oplus \overline{280}. \tag{36}$$

In view of this use of the notation 70 in table 9, a prime is used for $70' = (3, 0, 1)$ in table 8.

The energy eigenvalue of any state of the a_3 Fock space follows from (22) for $D = 15$, and can be read off table 8.

5.2. Remarks about higher order fermionic invariants

We may expect to find, for theories whose invariance algebra is a Lie algebra of rank 3 or higher, not only properties qualitatively like those seen in section 3 for rank 2 Lie algebras, but some additional ones. We begin by reviewing some material from [5].

For Lie algebras of rank higher than 2 new $\mathfrak{g}$-invariant fermionic operators enter the scene [5], such as

$$Q_5 = \tfrac{1}{5} \Omega_{ijklm} c_i c_j c_k c_l c_m \qquad Q_7 = \tfrac{1}{7} \mathrm{i} \Omega_{ijklmpq} c_i c_j c_k c_l c_m c_p c_q \tag{37}$$

Table 8. Data for a_3.

$d(\lambda, \mu, \nu)$	(λ, μ, ν)	$c_2(\lambda, \mu, \nu)$	E_q	$d(\lambda, \mu, \nu)$	(λ, μ, ν)	$c_2(\lambda, \mu, \nu)$	E_q
1	(0, 0, 0)	0	5	$85'$	(0, 6, 0)	$\frac{45}{8}$	
6	(0, 1, 0)	$\frac{5}{8}$		105	(0, 4, 0)	4	1
10	(2, 0, 0)	$\frac{9}{8}$		175	(1, 2,1)	3	2
15	(1, 0, 1)	1	$\frac{7}{2}$	189	(5, 0, 1)	5	
20	(0, 2, 0)	$\frac{3}{2}$	$\frac{7}{2}$	196	(0, 5, 0)	$\frac{45}{8}$	
35	(4, 0, 0)	3	2	256	(3, 1, 1)	$\frac{15}{4}$	$\frac{5}{4}$
45	(2, 1, 0)	2	3	270	(4, 0,2)	$\frac{37}{8}$	
50	(0, 3, 0)	$\frac{21}{8}$		280	(2, 3, 0)	$\frac{9}{2}$	$\frac{1}{2}$
64	(1, 1, 1)	$\frac{15}{8}$		300	(2, 1, 2)	$\frac{29}{8}$	
$70'$	(3, 0, 1)	$\frac{21}{8}$		$300'$	(3, 0, 3)	$\frac{9}{2}$	$\frac{1}{2}$
84	(2, 0, 2)	$\frac{5}{2}$	$\frac{5}{2}$	729	(2, 2, 2)	5	0

Table 9. $\mathcal{F}$ for a_3.

n	dim $ad^{\wedge n}$	1	15	20	70	84	90	105	175	$300'$	512	560	729
0(15)	1	1											
1(14)	15		1										
2(13)	105		1				1						
3(12)	455	1	1	1	1	1	1		1				
4(11)	1365		2	2	1	2	1	1	2		1		
5(10)	3003	1	2	1		2	3	1	3	1	2	1	
6(9)	5005		3	1	1	3	3		5	2	2	2	1
7(8)	6435	1	2	3	1	4	3	2	5	1	3	1	3

where Ω_5, and Ω_7 are totally antisymmetric *ad*-invariant tensors of ranks 5 and 7 [11, 12] and their adjoints. The available tensors correspond to the cohomology cocycles of the Lie algebras in question, so that we might also write $Q = Q_3$. However, Ω_5 occurs only for the a_n family, see, e.g., [13]. Only for a_n for $n > 3$ does Ω_5 enter our analysis non-trivially. While it is well defined for a_2, Hodge duality obviates the need for explicit use of it, and the same applies to Ω_7 for b_2 and Ω_{11} for $\mathfrak{g}_2$.

We need to consider various $\mathfrak{g}$-invariant spinorial quantities, using the notation

$$\begin{aligned}
q_{30} &= Q = \tfrac{1}{3}\mathrm{i}C_{ijk}c_ic_jc_k \\
q_{21} &= Q' = \tfrac{1}{2}\mathrm{i}C_{ijk}c_ic_j\pi_k \\
q_{12} &= Q'^{\dagger} = \tfrac{1}{2}\mathrm{i}C_{ijk}c_i\pi_j\pi_k \\
q_{03} &= Q^{\dagger} = \tfrac{1}{3}\mathrm{i}C_{ijk}\pi_i\pi_j\pi_k \\
q_{50} &= Q_5 = \tfrac{1}{5}\Omega_{ijklm}c_ic_jc_kc_lc_m \\
q_{41} &= \tfrac{1}{4}\Omega_{ijklm}c_ic_jc_kc_l\pi_m \\
q_{32} &= \tfrac{1}{6}\Omega_{ijklm}c_ic_jc_k\pi_l\pi_m .
\end{aligned} \tag{38}$$

From the standpoint of the partner theory $\mathcal{P}(D)$ of $\mathcal{S}(D)$, with Hamiltonian $H_p = K$ and natural supercharges q_{21} and q_{12}, one learns from analysis of possibilities drawn from (38) that q_{50} and q_{05} each anticommute with each of q_{21} and q_{12}, and so are hidden conserved supercharges. This result, given in [5], can be checked with the aid of generalized Jacobi identities to be found in [12]. A similar result [5] holds also for $q_{70} = Q_7$ and q_{07}.

From the standpoint of $\mathcal{S}(D)$ itself with Hamiltonian H_q and natural supercharges q_{30} and q_{03}, one finds

$$\{q_{30}, q_{0s}\} \neq 0 \qquad \{q_{03}, q_{50}\} \neq 0. \tag{39}$$

This does not cause inconsistency with $[Q_5, H_q] = 0$, which only requires that the operator of (39) commute with $Q^\dagger = q_{03}$, which it does.

The remarks of the previous paragraphs do not exhaust what can usefully be said. For Fock space considerations, one can employ a suitable set of anti-commuting $\mathfrak{g}$-invariant spinor operators, drawn from a list such as (38), to generate, from some state $|R, \phi\rangle$ transforming under $\mathfrak{g}$ according to some irrep R of $\mathfrak{g}$, states of higher fermion number which also transform according to R. In section 3, we used Q and Q' to construct doublets and quadruplets. For a_3, Q_5 enters the picture non-trivially. If we consider the set of spinorial invariants, all of which commute with H_q,

$$Q' = q_{21} \qquad Q = q_{30} \qquad Q_5 = q_{50} \tag{40}$$

an anticommuting set of operators which change fermion number by 1, 3, 5, we might expect to find supersymmetry related octuplets in the Fock space belonging to the same R. We note that q_{41} could be added to (40), but this does not appear to yield immediately useful additional information.

Thus, if $|\Phi\rangle$ is a Fock space state with fermion number n, then the set of eight states

$$|\Phi\rangle,\ Q'|\Phi\rangle,\ Q|\Phi\rangle,\ Q_5|\Phi\rangle,\ Q'Q|\Phi\rangle,\ Q'Q_5|\Phi\rangle,\ QQ_5|\Phi\rangle,\ QQ'Q_5|\Phi\rangle \tag{41}$$

should yield an octuplet of states of a given irrep R of a_3 with fermion numbers

$$n,\ (n+1),\ (n+3),\ (n+5),\ (n+4),\ (n+6),\ (n+8),\ (n+9). \tag{42}$$

A promising place in which we might search for such octuplets is found in the case of the irrep $15 = ad$. We can find quadruplets of 15's with the following fermion numbers:

$$\{1, 2, 4, 5\}, \quad \{3, 4, 6, 7\}, \quad \{5, 6, 8, 9\}, \quad \{6, 7, 9, 10\}, \quad \{8, 9, 11, 12\}, \quad \{10, 11, 13, 14\} \tag{43}$$

and may expect to connect these pairwise into octuplets, by action on the first three of Q_5, although supplying detail is probably out of the question. One can examine other possibilities using the data in table 7.

Another interesting set of invariants is $Q = Q_3, Q_5, Q_7$. Acting on the Fock vacuum, one finds this set of a_3 singlet states

$$|0\rangle,\ Q|0\rangle,\ Q_5|0\rangle,\ Q_7|0\rangle,\ QQ_5|0\rangle,\ QQ_7|0\rangle,\ Q_5Q_7|0\rangle,\ QQ_5Q_7|0\rangle \tag{44}$$

of respective fermion numbers 0, 3, 5, 7, 8, 10, 12, 15. These states correspond exactly to the terms of the Poincaré polynomial, see [13],

$$P(x) = (1 + x^3)(1 + x^5)(1 + x^7) \tag{45}$$

of a_3. Obviously this is a general result, valid in evident form for each $\mathfrak{g}$. We note $Q'|O\rangle = 0$, so that Q' is absent from (44).

Finally, one might ask what operator, which commutes with fermion number and with the action of a_2, accounts for the degeneracy of the two irreps 8 and 27 for $ad^{\wedge 4}$ for a_2. Similar questions arise for other $\mathfrak{g}$. In this context we can say the following. For a_2, the anticommutator of $Q = q_{30}$ and q_{14} has equal numbers of c_i and π_i and, with a factor i in the definition of Q and not in q_{14}, defines a Hermitian operator V

$$V = \{Q, q_{14}\} \propto \mathrm{i} c_i d_j \pi_k \pi_l \pi_m \Omega_{ijklm} \qquad d_i = c_{ijk} c_j c_k. \tag{46}$$

Using the methods and the notation of [8], this gives rise to

$$|8,4\rangle \propto V|8,4\rangle' \tag{47}$$

which may be placed alongside the results (28)–(30) noted in section 3.2.

6. Spectrum generating algebras

The spectrum generating algebra $\mathcal{A}(\mathcal{S})$ of a quantum mechanical system $\mathcal{S}$ is defined as follows. It is that Lie algebra $\mathcal{A}(\mathcal{S})$, or possibly Lie superalgebra, such that the vector space of states of $\mathcal{S}$ is the carrier space of a single irrep of $\mathcal{A}(\mathcal{S})$.

For the models $\mathcal{S}(D)$ of $\mathfrak{g}$-invariant systems of D fermions, $D = \dim \mathfrak{g}$, that are the subject of the present study, we have

$$\mathcal{A}(\mathcal{S}) = so(2D+1). \tag{48}$$

In fact the Fock space of $\mathcal{S}(D)$ coincides with the spinor representation of $so(2D+1)$, each one possessing the dimension 2^D.

To explain the origin of $so(2D+1)$ in our considerations, we employ the $2D$ fermionic variables $c_i, \pi_i, i \in \{1, 2, \ldots, D\}$, subject to (17). The largest Lie algebra that we can build with generators quadratic in these variables involves

$$[c_i, \pi_j], \quad c_i c_j, \quad \pi_i \pi_j \tag{49}$$

and these close on an algebra of dimension

$$D^2 + 2\left[\tfrac{1}{2}D(D-1)\right] = \dim so(2D). \tag{50}$$

Also the subspaces of the Fock space of states of even and of odd fermion number are seen to be carrier spaces for the two spinor irreps of $so(2D)$. Further the set (49) of generators of $so(2D)$ reflect the symmetric Lie algebra structure that is typical of a Hermitian symmetric space, in this case $SO(2D)/U(n)$. Explicitly, we have

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{n}_+ + \mathfrak{n}_- \tag{51}$$

where the entries in (49) are the generators of the three subalgebras taken in the same order. It is obvious that $\mathfrak{h} = u(n)$, that $\mathfrak{n}_\pm$ are Abelian and that $[\mathfrak{n}_+, \mathfrak{n}_-] \cap \mathfrak{h}$.

To reach $\mathcal{A}(\mathcal{S}) = so(2D+1)$, requires us to add, to our set c_i, π_i of basic variables, a single Majorana fermion variable c such that

$$c = c^\dagger \qquad c^2 = 1 \qquad \{c, c_i\} = 0 \qquad \{c, \pi_i\} = 0. \tag{52}$$

Then the $2D = \dim so(2D+1) - \dim so(2D)$ generators needed to complete our realization of $so(2D+1)$ are

$$cc_i \qquad \text{and} \qquad c\pi_i. \tag{53}$$

As long as we consider a single system $\mathcal{S}(D)$, the role of c is, for most purposes—but see [14]—trivial. However if one wishes to consider a composite system of two independent realizations of $\mathcal{S}$, the fermionic nature of the single Majorana fermions contained in each one, needs to be fully respected to achieve a consistent formalism.

For a single $\mathcal{S}$ putting $c = 1$ we see that the generators (53) change the fermion numbers of states by ± 1, so that the Fock space carries an irrep of $so(2D+1)$. It is the spinor irrep of dimension 2^D.

This is consistent with reduction back to the $so(2D)$ view, since the spinor irrep of $so(2D+1)$ decomposes into the two spinor irreps of $so(2D)$.

It is clear that $so(2D+1)$ is not an invariance algebra of $\mathcal{S}(D)$. How then is the actual invariance algebra $\mathfrak{g}$, $\dim\mathfrak{g} = D$ related to (embedded in) $so(2D+1)$? It is generated by the subset

$$X_i = -\mathrm{i} c_{ijk} c_j \pi_k \qquad i, j \in \{1, 2, \ldots, D\} \tag{54}$$

of the generators (49) of $so(2D)$. The remaining generators linear in each of the c_i and π_i can be separated into sets which transform under $\mathfrak{g}$ according to each of the remaining constituents of the representation $ad \otimes ad$ of $\mathfrak{g}$. These however cannot be expected to commute with H_q.

7. Decomposition of $ad^{\wedge n}$ into irreps

For the purpose of assembling the data in tables 3, 5, 7 and 9, it is probably sufficient to state that these contain output from a C++ program designed to produce Lie algebra data such as this, and the Casimir eigenvalues needed in tables 2, 4, 6 and 8. In every case treated the results presented conform to the results of section 4 about the ground states of the models $\mathcal{S}(D)$. However, to understand the latter results as general results, valid for each simple compact $\mathfrak{g}$, requires us to present some detailed analysis of $ad^{\wedge n}$ for $\mathfrak{g}$. The purpose of this section then is to carry this analysis just far enough to explain, clearly and in general, the emergence of the results relating to the irrep $(2, \ldots, 2)$ of $\mathfrak{g}$.

A full treatment of the decompositions of $ad^{\wedge n}$, $n = 0, 1, 2, \ldots, D = \dim\mathfrak{g}$ into irreps of (simple compact) $\mathfrak{g}$ can be based on a generating function for the characters of $ad^{\wedge n}$.

Let l be the rank of $\mathfrak{g}$, and $p = (D-l)/2$ be the number of positive roots.

Let $\chi_{[n]}(g) = \mathrm{Tr}\,\Gamma(g)^{\wedge n}$ be the character of the totally antisymmetrized n-fold product of a $k \times k$ unitary matrix representation $g \mapsto \Gamma(g)$ of a compact simple Lie group. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of $\Gamma(g)$. Then we have the generating function

$$\sum_{n=0}^{k} x^n \chi_{[n]}(g) = \prod_{j=1}^{k} (1 + x\lambda_j) \qquad \chi_{[n]}(g) = \mathrm{Tr}\,\Gamma(g)^{\wedge n}. \tag{55}$$

Results of this type originate in the work of Molien [15]. A proof in the totally symmetrized case can be found in [16], see p 93 and p 203. Adaptation of this proof to the present case is easy.

In the case of $\Gamma = Ad$, where Ad, as an irrep of the Lie group G, is related by exponentiation to the irrep ad of the Lie algebra $\mathfrak{g}$ of G, we have

$$\chi_{ad}(g) = \mathrm{Tr}\,\Gamma(g) = \mathrm{Tr}\exp\mathrm{i}\sum_j a_j H_j = \sum_{\mathrm{r}} \exp\mathrm{i}\mathbf{a}\cdot\mathbf{r} \tag{56}$$

where the H_j are the matrices for ad of the Cartan generators of $\mathfrak{g}$, and the second sum is over all the weights of ad, i.e. over the roots of $\mathfrak{g}$, including the l null ones.

Suppose that the expansion of a root $\mathbf{r}$ of $\mathfrak{g}$ with respect to the simple roots $\boldsymbol{\alpha}_j$, $j = 1, 2, \ldots, l$ of $\mathfrak{g}$ reads as

$$\mathbf{r} = \sum_{j=1}^{l} x_j \boldsymbol{\alpha}_j \tag{57}$$

where the x_j are non-negative integers for positive $\mathbf{r}$, and that $\mathbf{a}$ has components c_j, $j = 1, 2 \ldots, l$, with respect to the dual basis. Then it follows that

$$\exp\mathrm{i}\mathbf{a}\cdot\mathbf{r} = \prod_{j=1}^{l} \exp\mathrm{i}c_j x_j = \prod_{j=1}^{l} t_j^{\,x_j} \qquad t_j = \exp\mathrm{i}c_j \tag{58}$$

and hence that

$$\sum_{n=0}^{D} x^n \chi_{[n]}(t_1, \ldots, t_l) = \prod_{\text{all roots}} \left(1 + x \prod_{j=1}^{l} t_j^{\,x_j} \right). \tag{59}$$

The product has one factor for each positive and each negative root, and the null root contributes a factor $(1+x)^l$ to the right-hand side of (59). Equation (59) provides the required generating function.

The coefficient of x gives back again the expression

$$\chi_{[1]} = \chi_{ad} = \sum_{\text{all roots}} \left(\prod_{j=1}^{l} t_j^{\,x_j} \right) = \sum_{\text{all roots}} \exp \mathrm{i}\mathbf{a} \cdot \mathbf{r}. \tag{60}$$

The highest root $\mathbf{r}_p$ of $\mathfrak{g}$ features here as the highest weight ad.

From (59), it can be seen that, for each n, $1 \leqslant n \leqslant p$, the highest weight of $\mathfrak{g}$ contained in $\chi_{[n]}$ must be the sum of some set of n *distinct* positive roots of $\mathfrak{g}$. For the (well-understood) case of $n = 2$, there is, for all $\mathfrak{g}$ except a_l, a partial ordering of roots by height of the sort $\mathbf{r}_p > \mathbf{r}_{p-1} > \mathbf{r}_{p-2}$ so that there is in $ad^{\wedge 2}$ an irrep of $\mathfrak{g}$ of easily identified highest weight equal to $\mathbf{r}_p + \mathbf{r}_{p-1}$. For $n = 2$ and a_l, the corresponding partial ordering reads $\mathbf{r}_p > \mathbf{r}_{p-1}, \mathbf{r}_{p-2} > \mathbf{r}_{p-3}$, and we are led to a complex conjugate pair of irreps of a_l of highest weights $\mathbf{r}_p + \mathbf{r}_{p-1}$ and $\mathbf{r}_p + \mathbf{r}_{p-2}$.

There is an algorithm available [17] for each n, $1 \leqslant n \leqslant p$, for finding suitable sums of n distinct roots of $\mathfrak{g}$ that lead to the identification of certain 'leading' irreps of $\mathfrak{g}$ contained in the decomposition of the corresponding $ad^{\wedge n}$. However, for the limited purpose of explaining the key result of section 4, we do not need to develop this. In the special and relevant case of $n = p$, it follows from (59) that the highest weight of $ad^{\wedge p}$ is equal to the sum of all the positive roots of $\mathfrak{g}$, the unique way of taking the sum of distinct positive roots when $n = p$. In other words, in the $n = p$ term of (59), we use the second term of each positive root factor, and the trivial term of the null and negative root factors. Since the sum of all the positive roots of $\mathfrak{g}$ is equal to 2δ, it follows that the irrep of this highest weight, $2\delta = (2, \ldots, 2)$ in Dynkin notation, is contained in $ad^{\wedge p}$ for any $\mathfrak{g}$. This is the result that section 4 needs. Much more can be said in the general area in question here, and we refer to [17] for this.

Going on to $ad^{\wedge(p+r)}$, $r \in \{1, 2, \ldots, l\}$, it is clear that no terms of weight greater than 2δ can be found in $\chi_{[p+r]}$, and the best that can be done is to use the x-terms of r of the factors $(1+x)^l$ of (59). It follows that the irrep of highest weight 2δ occurs $\binom{l}{r}$ in $ad^{\wedge(p+r)}$ times for each $\mathfrak{g}$, and hence 2^l times in all, as noted in section 1. Going beyond $n = p + l$ brings the negative weights into the picture, but it is not necessary to dwell on this: it is obvious that $ad^{\wedge n}$ and $ad^{\wedge(D-n)}$ are equivalent.

It is straightforward to use (59) to deduce, for small enough $\mathfrak{g}$ (which covers all the cases discussed here) the complete decomposition into irreps of $ad^{\wedge n}$ for all n. Assuming initially that one knows all the relevant characters, one notes the highest weight that occurs in $\chi_{[n]}$, and subtracts from $\chi_{[n]}$ the character of the corresponding irrep. One next finds the highest weight term of the remainder of $\chi_{[n]}$, and again subtracts the character of the corresponding irrep, and so on until there is no remainder. In fact, the required characters can, for $n = 1, 2, \ldots,$ be systematically found as they are needed. One starts with the knowledge (60) of χ_{ad}, and perhaps also the easily found character of the defining irrep of $\mathfrak{g}$. When one first needs a new character, one seeks, and can always find, a direct product of irreps, such that the character of each factor is known, as are all the characters in the decomposition except that of the one sought. This process was carried through for the three rank two Lie algebras and for a_3, ahead of confirmation of the results by C++ program. It is especially tractable for the rank 2 algebras where Speiser's rule [18] makes it easy to decompose any twofold direct product graphically into irreps.

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